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LETTER TO THE EDITOR

Deformation of orthosymplectic Lie superalgebra $osp(1|4)$

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Abstract

Triangular deformation of the orthosymplectic Lie superalgebra $osp(1|4)$ is defined by chains of twists. The corresponding classical r-matrix is obtained by a contraction procedure from the trigonometric r-matrix. The carrier space of the constant r-matrix and the twist element is the Borel subalgebra. The known super-Jordanian twist of $osp(1|2)$ is generalized as an extended twist to the orthosymplectic Lie superalgebras of higher rank. The dimension of the cobracket kernel is related to the number of generators with primitive twisted coproduct of the deformed algebra.

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Orthosymplectic Lie superalgebras $osp(m|2n)$ have a variety of applications in gauged supergravity, supersymmetric quantum mechanics, integrable $\mathbb{Z}_2$-graded spin chains etc. Similar applications with corresponding changes can have quantum deformations of these superalgebras. In this letter the known super-Jordanian deformation of $osp(1|2)$ is extended to the case of the orthosymplectic Lie superalgebras of higher rank, and for $g = osp(1|4)$ is given by explicit construction of a universal twist element $\mathcal{F} = \sum_i f_i^{(1)} \otimes f_i^{(2)} \in \mathcal{U}(g) \otimes \mathcal{U}(g)$ [1]. It is a natural extension of known twist for $osp(1|2)$ [2, 3] along the lines presented in [4] about twists with deformed carrier subspaces. The constructed twist includes all the generators of the Borel subalgebra $B_+ \subset osp(1|4)$. The r-matrix with the Borel subalgebra as the carrier space is also given for $osp(1|2n)$.

Deformed quantum superalgebras can be used to define a noncommutative version of the super anti-de Sitter space (s-AdS) generalizing [5–8], deformation of superconformal quantum mechanics [9], deformed spin chains [10], etc.

Triangular deformation of the universal enveloping algebra $\mathcal{U}(g)$ of a Lie algebra g is defined by a twist element $\mathcal{F} \in \mathcal{U}(g) \otimes \mathcal{U}(g)$ [1]. The latter transforms maps of coproduct

$\Delta \to \Delta^{(t)}$ and antipode $S \to S^{(t)}$ of the universal enveloping algebra (UEA)

$$\Delta^{(t)}(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1} \qquad S^{(t)} = uSu^{-1} \quad x \in \mathcal{U}(g) \tag{1}$$

where $u = \sum_i f_i^{(1)} S\big(f_i^{(2)}\big) \in \mathcal{U}(g)$. The universal R-matrix is connected with the classical r-matrix

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = 1 + \xi r + \mathcal{O}(\xi 2) \qquad \mathcal{F}_{21} = \sum_i f_i^{(2)} \otimes f_i^{(1)}$$

where ξ is the quasi-classical parameter.

The direction of twist deformation is given by a classical triangular r-matrix $r \in g \wedge g$, which is a solution of the classical (super) Yang–Baxter equation (cYBE) on $g \otimes g \otimes g$ [11]

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0 \tag{2}$$

where in the super-case the commutators and the tensor products are $\mathbb{Z}_2$ graded:

$$[a, b] = ab - (-1)^{p(a)p(b)}ba \qquad (a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd).$$

The parity of a homogeneous element b of a $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$ is denoted by $p(b)$. For basic classical Lie superalgebras the classical r-matrix we are looking for, can be obtained by a contraction procedure from the trigonometric r-matrix

$$r(\lambda - \mu) = \big(r_0\, \mathrm{e}^{\lambda-\mu} + r_0^{21}\big)\big/(\mathrm{e}^{\lambda-\mu} - 1) \tag{3}$$

where r_0 is the super-analogue of the Drinfeld–Jimbo constant r-matrix, the solution to the cYBE,

$$r_0 = \frac{1}{2}\sum k_i \otimes k_i + \sum_{\alpha\in\Delta_+} e_\alpha \otimes e_{-\alpha}. \tag{4}$$

Here $\{k_i\}$ is an orthonormal basis of the Cartan subalgebra, and Δ_+ is the set of positive (even and odd) roots. The sum

$$c_2^{\otimes} := r_0 + r_0^{21} = \sum k_i \otimes k_i + \sum_{\alpha\in\Delta_+} (e_\alpha \otimes e_{-\alpha} + (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha)$$

is an element of $\mathcal{U}(g) \otimes \mathcal{U}(g)$ (the tensor Casimir element) invariant with respect to the adjoint action:

$$\big[x \otimes 1 + 1 \otimes x, c_2^{\otimes}\big] = 0 \qquad x \in g.$$

Take the long root generator e_θ and consider adjoint transformation by the group element $\exp(te_\theta)$:

$$Ad(\exp(te_\theta))^{\otimes 2} r(\lambda - \mu) = \frac{1}{2}\left(\coth\frac{\lambda-\mu}{2}\cdot c_2^{\otimes} + \sum_{\alpha\in\Delta_+}(e_\alpha \otimes e_{-\alpha} - (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha)\right.$$
$$\left. + t\sum_{\alpha\in\Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}]\right). \tag{5}$$

Scaling the spectral parameter $\lambda \to \varepsilon\lambda$, $\mu \to \varepsilon\mu$ and $t \to 2t/\varepsilon$, one gets after contraction

$$\lim_{\varepsilon\to 0} \varepsilon Ad(\exp(te_\theta/\varepsilon))^{\otimes 2} r(\varepsilon(\lambda - \mu)) = \frac{c_2^{\otimes}}{(\lambda-\mu)} + t\sum_{\alpha\in\Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}]. \tag{6}$$

Due to the Lie algebra nature of the cYBE (2), the resulting expression satisfies this equation as well and both terms satisfy it separately. According to the long root θ property, all the generators in the constant r-matrix (6) correspond to positive roots, and one generator belongs to the Cartan subalgebra $[e_\theta, e_{-\theta}]$.

To deform the UEA $\mathcal{U}(osp(1|4))$ the second term of the r-matrix (6) will be used. The Cartan–Weyl basis of the orthosymplectic Lie superalgebra $osp(1|4)$ has the following generators: H, J (the Cartan subalgebra), $v_\pm, w_\pm$ (the odd root generators), $X_\pm = \pm(v_\pm)^2, Y_\pm = \pm(w_\pm)^2$, $U_\pm, Z_\pm$. There are two $osp(1|2)$ subalgebras $\{H, v_\pm, X_\pm\}$ and $\{J, w_\pm, Y_\pm\}$, and the even root generators define the Lie subalgebra $sp(4) \simeq so(5)$. Constructing a universal twist element $\mathcal{F}$ we shall use only commutators of the Borel subalgebra generators with subscripts +. Some of these commutation relations are

$$[H, X_+] = 2X_+ \qquad [H, v_+] = v_+ \qquad [H, U_+] = U_+ \tag{7}$$

$$[H, Z_+] = Z_+ \qquad [Z_+, U_+] = 2X_+ \qquad [Z_+, Y_+] = U_+ \tag{8}$$

$$[Z_+, w_+] = v_+ \qquad [v_+, w_+] = U_+. \tag{9}$$

The generators Z_+, w_+ correspond to the simple roots α_1, α_2 of $osp(1|4)$. The long root $\theta = 2(\alpha_1 + \alpha_2)$ and the corresponding generator $e_\theta = X_+$ commute with all positive root generators and with the generator J of the Cartan subalgebra. In the case of $osp(1|4)$, the constant term in the r-matrix (6) is ($[X_+, X_-] = H$)

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+. \tag{10}$$

This is an extension by the odd generator v_+ of the classical r-matrix defining the extended Jordanian twisting of the $\mathcal{U}(so(5)) \simeq \mathcal{U}(sp(4))$ [4, 12].

A solution to the cYBE defines a cobracket δ on the Lie (super) algebra: a map $\delta : g \to g \wedge g$,

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r] \qquad x \in g.$$

The cobracket δ is related to the twisted coproduct $\Delta^{(t)}$ (1) through the quasi-classical limit

$$\delta(x) = \lim_{\xi \to 0} \frac{\Delta^{(t)}(x) - \Delta^{(t)}_{\text{op}}(x)}{\xi}$$

where the opposite coproduct is $\Delta^{(t)}_{\text{op}}(x) = \sum x_{(2)} \otimes x_{(1)}$ if $\Delta^{(t)}(x) = \sum x_{(1)} \otimes x_{(2)}$.

It is important to point out that the dimension of the kernel of δ is equal to the number of independent primitive elements in the quantum $\mathcal{U}_\xi(g)$ which is a quantization of the bialgebra $\mathcal{U}(g)$. A proof can be given by transition to the dual Poisson–Lie group using the quantum duality principle [13, 14].

The kernel of the cobracket defined by the r-matrix (10) is generated by the elements X_+, J, Y_+, w_+. It is easy to see that the kernel of the cobracket δ is a Lie subalgebra $\operatorname{Ker}\delta \subset g$. If there is a triangular r-matrix with its carrier in $\operatorname{Ker}\delta$, then the sum of this additional r-matrix and the one defining δ satisfies the cYBE as well. In this case we are interested in there is a super-Jordanian r-matrix [3] in $\operatorname{Ker}\delta$

$$r^{(sj)} = J \wedge Y_+ - w_+ \wedge w_+. \tag{11}$$

Finally, we shall construct a universal twist corresponding to the r-matrix:

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+ + J \wedge Y_+ - w_+ \otimes w_+. \tag{12}$$

(One can also add an Abelian term $X_+ \wedge Y_+$, and introduce a few independent parameters in front of different constituent r-matrices by rescaling of appropriate generators or by a similarity transformation.)

This form of the classical r-matrix can be generalized to the case of the universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1|2n)$. It has (positive) even simple root generators $Z_k, k = 1, 2, \ldots, n-1, \alpha_k = \varepsilon_k - \varepsilon_{k+1}$, and one odd simple root generator v_n, $\alpha_n = \varepsilon_n$. Other positive root generators are $X_i = v2_i$, $i = 1, 2, \ldots, n$, Z_{kj} and U_{kj} related to positive roots $2\varepsilon_i$, $\varepsilon_k - \varepsilon_j$ and $\varepsilon_k + \varepsilon_j$, $1 \leqslant k < j \leqslant n$, where ε_i are the Cartesian unit vectors [15]. Commutation relations

$$[H_k, X_k] = 2X_k \qquad [H_k, v_k] = v_k \tag{13}$$

$$[Z_{kj}, U_{kj}] = 2X_k \qquad [v_k, v_j] = U_{kj} \tag{14}$$

$$[Z_{kj}, v_j] = v_k \qquad [H_k, Z_{kj}] = Z_{kj} \qquad [H_k, U_{kj}] = U_{kj} \tag{15}$$

are used to prove that the following expression with arbitrary parameters a_k

$$r = \sum_{k=1}^{n} a_k \left(H_k \wedge X_k + \sum_{j>k}^{n} Z_{kj} \wedge U_{kj} - v_k \otimes v_k \right) \tag{16}$$

satisfies the cYBE. Similar to the symplectic algebra $sp(2n)$ [18] the sum of r-matrices (16) corresponds to the sequence of injections

$$osp(1|2) \subset osp(1|4) \subset \cdots \subset osp(1|2n).$$

The construction of an explicit form of the twist corresponding to the triangular r-matrix (10) will be realized according to the recipe known as chains of twists [16, 17]. The classical r-matrix is decomposed into a few terms which generate factors of the universal twist. This decomposition consists of the Jordanian r-matrix

$$r^{(j)} = H \otimes X_+ - X_+ \otimes H := H \wedge X_+ \tag{17}$$

extended Jordanian $r^{(ej)}$ and super-Jordanian r-matrices

$$r^{(ej)} = r^{(j)} + Z_+ \wedge U_+ \qquad r^{(sj)} = r^{(j)} - v_+ \otimes v_+. \tag{18}$$

Twist elements corresponding to $r^{(ej)}$ and $r^{(sj)}$ are known [16, 3], and they are represented in a factorized form

$$\mathcal{F}^{(ej)} = \mathcal{F}^{(e)}\mathcal{F}^{(j)} \qquad \mathcal{F}^{(sj)} = \mathcal{F}^{(s)}\mathcal{F}^{(j)} \tag{19}$$

where

$$\mathcal{F}^{(j)} = \exp(H \otimes \sigma) \qquad \mathcal{F}^{(e)} = \exp\left(\tfrac{1}{2} Z_+ \otimes U_+ e^{-\sigma}\right) \tag{20}$$

$$\mathcal{F}^{(s)} = \exp(-v_+ \otimes v_+ f(\sigma \otimes 1, 1 \otimes \sigma)) \qquad \sigma = \tfrac{1}{2}\ln(1 + X_+). \tag{21}$$

The function f is symmetric. It can be presented as the series expansion $\sum f_n$ [3], or through the factorization of $\mathcal{F}^{(s)}$ as

$$\mathcal{F}^{(s)} = (1 - (v_+ \otimes v_+)(f_1(\sigma) \otimes f_1(\sigma)))\mathcal{F}^{(c)} \qquad f_1(\sigma) = (\mathrm{e}^{\sigma} + 1)^{-1} \tag{22}$$

with an appropriate coboundary twist [9]

$$\mathcal{F}^{(c)} = (u \otimes u)\Delta(u^{-1}) \qquad u = \left(\tfrac{1}{2}(\mathrm{e}^{\sigma} + 1)\right)^{\frac{1}{2}}.$$

Due to the commutativity of X_+ or σ and v_+ with Z_+, U_+, the twisting by $\mathcal{F}^{(s)}$ after $\mathcal{F}^{(j)}$ does not change the coproducts of Z_+ and U_+, and vice versa. Hence, one can arrange these twists to form an extended super-Jordanian twist

$$\mathcal{F}^{(esj)} = \mathcal{F}^{(s)}\mathcal{F}^{(e)}\mathcal{F}^{(j)} = \mathcal{F}^{(e)}\mathcal{F}^{(s)}\mathcal{F}^{(j)} \tag{23}$$

corresponding to the r-matrix (10).

To take into account a further extension of the r-matrix (12) by a super-Jordanian term of the second $osp(1|2)$ subalgebra, we have to use a deformed carrier space transforming the generators Y_+ and w_+ according to [4, 16].

The generators v_+, U_+ and $\sigma(X_+) = \frac{1}{2}\ln(1+X_+)$ enter this transformation. After the twist (23) their coproducts are

$$\Delta^{(esj)}(\sigma) := \mathcal{F}^{(esj)}\Delta(\sigma)(\mathcal{F}^{(esj)})^{-1} = \Delta_0(\sigma) := \sigma\otimes 1 + 1\otimes\sigma$$
$$\Delta^{(esj)}(v_+) = \Delta^{(sj)}(v_+) = v_+\otimes 1 + \mathrm{e}^{\sigma}\otimes v_+$$
$$\Delta^{(esj)}(U_+) = \Delta^{(ej)}(U_+) = U_+\otimes \mathrm{e}^{\sigma} + \mathrm{e}^{2\sigma}\otimes U_+.$$

The twisted coproducts of the Borel subalgebra generators J, Y_+, w_+ of the second $osp(1|2)$ are

$$\Delta^{(esj)}(J) = \mathcal{F}^{(esj)}\Delta(J)(\mathcal{F}^{(esj)})^{-1} = \Delta_0(J) := J\otimes 1 + 1\otimes J$$
$$\Delta^{(esj)}(Y_+) = \Delta_0(Y_+) + \tfrac{1}{2}U_+\otimes U_+\,\mathrm{e}^{-\sigma} + \tfrac{1}{4}X_+\otimes U_+^2\,\mathrm{e}^{-2\sigma}$$

and we get a rather lengthy expression for $\Delta^{(esj)}(w_+)$.

The deformed generator $\widetilde{Y}_+$ with the primitive coproduct is given by the adjoint transformation [4, 18]

$$\widetilde{Y}_+ = Ad\exp(-Z_+U_+\sigma/2X_+)Y_+ = Y_+ - \tfrac{1}{4}U2_+\,\mathrm{e}^{-2\sigma}. \qquad (24)$$

Hence, the second Jordanian factor is similar to the $sp(4)\simeq so(5)$ case [4] and has the form $\exp(J\otimes\sigma(\widetilde{Y}_+))$. Although, the deformed coproduct of the odd generator w_+ is rather cumbersome after the transformation (24) the coproduct of $\widetilde{w}_+$

$$\widetilde{w}_+ = Ad\exp(-Z_+U_+\sigma/2X_+)w_+ = w_+ - \frac{1}{2}v_+U_+\frac{\mathrm{e}^{-\sigma}}{\mathrm{e}^{\sigma}+1}$$

is primitive as well. Hence, one can add to (23) an additional factor $\mathcal{F}^{(sj2)}$ corresponding to the second $osp(1|2)$

$$\mathcal{F}^{(sj2)} = \exp(-\widetilde{w}_+\otimes\widetilde{w}_+ f(\widetilde{\sigma}\otimes 1, 1\otimes\widetilde{\sigma}))\exp(J\otimes\sigma(\widetilde{Y}_+))$$

where $\widetilde{\sigma} := \sigma(\widetilde{Y}_+) = \frac{1}{2}\ln(1+\widetilde{Y}_+)$.

A universal twist with the carrier space including all generators of the Borel subalgebra $B_+\subset osp(1|4)$ is given by the product

$$\mathcal{F} = \mathcal{F}^{(sj2)}\mathcal{F}^{(esj)} = \mathcal{F}^{(sj2)}\mathcal{F}^{(s)}\mathcal{F}^{(e)}\mathcal{F}^{(j)} \qquad (25)$$

where the super-Jordanian twist $\mathcal{F}^{(sj2)}$ [3] is constructed using the generators $J, \widetilde{Y}_+, \widetilde{w}_+$ as in (18), (20). The universal R-matrix [1] of the twisted $\mathcal{U}(osp(1|4))$ is $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$.

A chain of twists similar to (12) can be constructed for $\mathcal{U}(osp(1|2n))$ starting with the r-matrix (16) as it was in the case of $\mathcal{U}(sp(2n))$ and $\mathcal{U}(so(2n+1))$ [18, 19]. This would extend in some sense an analogy between algebras $so(2n+1)$ and $osp(1|2n)$ [20].

One can introduce a more convenient set of generators of the twisted Borel subalgebra $B_+\subset osp(1|4)$ using the FRT-formalism [21], and the upper triangular form of the generators of B_+ in the defining 5×5 irreducible representation ρ. The new generators are the entries of the universal L-operator: $L = (\rho\otimes\mathrm{id})\mathcal{R}$. Using the symmetric grading $(0,0,1,0,0)$ of the rows and columns, one gets

$$L = \begin{pmatrix} T^{-1} & u & V & z & h \\ 0 & K^{-1} & W & j & \tilde{z} \\ 0 & 0 & 1 & \widetilde{W} & \widetilde{V} \\ 0 & 0 & 0 & K & \tilde{u} \\ 0 & 0 & 0 & 0 & T \end{pmatrix}.$$

The coproduct of these generators is given by the matrix product of the L-operators. The commutation relations follow from the RTT-relation: $RL_1L_2 = L_2L_1R$ [21], taking into account extra signs in the Z_2-graded tensor product [11]. The R-matrix can be obtained using also a super-version of the $r3 = 0$ theorem: $R = \exp(\eta r_\rho)$, where $r_\rho = (\rho \otimes \rho)r$. Similarly to the twisting of $osp(1|2)$ [3] one can express the new generators in terms of the initial ones.

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